Exercise 2.5.9

Solve Laplace's equation inside a 90° sector of a circular annulus ($a < r < b$, $0 < \theta < \pi/2$) subject to the boundary conditions [*Hint*: In polar coordinates,

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

it is known that if $u(r,\theta) = \phi(\theta)G(r)$, then $\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2}$.]:

(a) $u(r,0) = 0,\qquad u(r,\pi/2) = 0,\qquad u(a,\theta) = 0,\qquad u(b,\theta) = f(\theta)$

(b) $u(r,0) = 0,\qquad u(r,\pi/2) = f(r),\qquad u(a,\theta) = 0,\qquad u(b,\theta) = 0$

Solution

Because the Laplace equation is linear and homogeneous, the method of separation of variables can be applied to solve it. Assume a product solution of the form $u(r,\theta) = R(r)\Theta(\theta)$ and plug it into the PDE.

$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} &= 0 \\
\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}R(r)\Theta(\theta)\right] + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}R(r)\Theta(\theta) &= 0 \\
\frac{\Theta(\theta)}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{R(r)}{r^2}\frac{d^2\Theta}{d\theta^2} &= 0
\end{aligned}$$

Multiply both sides by $r^2/[R(r)\Theta(\theta)]$ in order to separate variables.

$$\frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\Theta(\theta)}\frac{d^2\Theta}{d\theta^2} = 0$$

$$\frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\frac{1}{\Theta(\theta)}\frac{d^2\Theta}{d\theta^2}$$

The only way a function of r can be equal to a function of θ is if both are equal to a constant λ.

$$\frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\frac{1}{\Theta(\theta)}\frac{d^2\Theta}{d\theta^2} = \lambda$$

As a result of separating variables, the PDE has reduced to two ODEs—one in each independent variable.

$$\left.\begin{aligned}
\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) &= \lambda \\
-\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} &= \lambda
\end{aligned}\right\}$$

Values of λ for which nontrivial solutions to these ODEs and the associated boundary conditions exist are called eigenvalues, and the solutions themselves are called eigenfunctions. Note that it doesn't matter whether the minus sign is grouped with r or θ as long as all eigenvalues are taken into account.

Part (a)

Substitute the product solution into the homogeneous boundary conditions.

$$\begin{array}{lllll} u(r,0)=0 & \to & R(r)\Theta(0)=0 & \to & \Theta(0)=0 \\ u\left(r,\frac{\pi}{2}\right)=0 & \to & R(r)\Theta\left(\frac{\pi}{2}\right)=0 & \to & \Theta\left(\frac{\pi}{2}\right)=0 \\ u(a,\theta)=0 & \to & R(a)\Theta(\theta)=0 & \to & R(a)=0 \end{array}$$

Solve the ODE for Θ.

$$\Theta''=-\lambda\Theta$$

Check for positive eigenvalues: $\lambda=\mu^2$.

$$\Theta''=-\mu^2\Theta$$

The general solution can be written in terms of sine and cosine.

$$\Theta(\theta)=C_1\cos\mu\theta+C_2\sin\mu\theta$$

Apply the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} \Theta(0)&=C_1=0 \\ \Theta\left(\frac{\pi}{2}\right)&=C_1\cos\mu\frac{\pi}{2}+C_2\sin\mu\frac{\pi}{2}=0 \end{aligned}$$

The second equation reduces to $C_2\sin\mu\frac{\pi}{2}=0$. To avoid the trivial solution, we insist that $C_2\neq 0$.

$$\begin{aligned} \sin\mu\frac{\pi}{2}&=0 \\ \mu\frac{\pi}{2}&=n\pi, \quad n=1,2,\ldots \\ \mu&=2n \end{aligned}$$

There are positive eigenvalues $\lambda=(2n)^2$, and the eigenfunctions associated with them are

$$\Theta(\theta)=C_2\sin\mu\theta \quad \to \quad \Theta_n(\theta)=\sin 2n\theta.$$

Note that this is only for positive integers because $n=0$ would lead to $\lambda=0$, and negative integers would lead to redundant values for λ. With $\lambda=4n^2$, solve the ODE for R now.

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right)=4n^2$$

Expand the left side.

$$\frac{r}{R}(R'+rR'')=4n^2$$

Multiply both sides by R and bring all terms to the left side.

$$r^2R''+rR'-4n^2R=0$$

This is an equidimensional ODE, so it has solutions of the form $R(r)=r^m$.

$$R=r^m \quad \to \quad R'=mr^{m-1} \quad \to \quad R''=m(m-1)r^{m-2}$$

Substitute these formulas into the ODE and solve the resulting equation for m.

$$r^2 m(m-1)r^{m-2} + rmr^{m-1} - 4n^2 r^m = 0$$

$$m(m-1)r^m + mr^m - 4n^2 r^m = 0$$

$$m(m-1) + m - 4n^2 = 0$$

$$m^2 - 4n^2 = 0$$

$$(m+2n)(m-2n) = 0$$

$$m = \{-2n, 2n\}$$

Two solutions to the ODE are $R = r^{-2n}$ and $R = r^{2n}$. By the principle of superposition, the general solution for R is a linear combination of these two.

$$R(r) = Ar^{-2n} + Br^{2n}$$

Apply the boundary condition at $r = a$ to determine one of the constants.

$$R(a) = Aa^{-2n} + Ba^{2n} = 0 \quad \rightarrow \quad A = -Ba^{4n}$$

This makes the R-eigenfunction

$$\begin{aligned} R(r) &= -Ba^{4n}r^{-2n} + Br^{2n} \\ &= Ba^{2n}\left(-\frac{a^{2n}}{r^{2n}} + \frac{r^{2n}}{a^{2n}}\right) \quad \rightarrow \quad R_n(r) = \frac{r^{2n}}{a^{2n}} - \frac{a^{2n}}{r^{2n}}. \end{aligned}$$

Check to see if zero is an eigenvalue: $\lambda = 0$.

$$\Theta'' = 0$$

The general solution is a straight line.

$$\Theta(\theta) = C_3\theta + C_4$$

Apply the boundary conditions to determine C_3 and C_4.

$$\begin{aligned} \Theta(0) &= C_4 = 0 \\ \Theta\left(\frac{\pi}{2}\right) &= C_3\frac{\pi}{2} + C_4 = 0 \end{aligned}$$

With $C_4 = 0$, the second equation reduces to $C_3\frac{\pi}{2} = 0$, which means $C_3 = 0$.

$$\Theta(\theta) = 0$$

The trivial solution is obtained, so zero is not an eigenvalue. Check to see if there are negative eigenvalues: $\lambda = -\gamma^2$.

$$\Theta'' = \gamma^2\Theta$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$\Theta(\theta) = C_5 \cosh\gamma\theta + C_6 \sinh\gamma\theta$$

Apply the boundary conditions to determine C_5 and C_6.

$$\Theta(0) = C_5 = 0$$
$$\Theta\left(\frac{\pi}{2}\right) = C_5 \cosh \gamma\frac{\pi}{2} + C_6 \sinh \gamma\frac{\pi}{2} = 0$$

With $C_5 = 0$, the second equation reduces to $C_6 \sinh \gamma\frac{\pi}{2} = 0$. No nonzero value of γ can satisfy this equation, so $C_6 = 0$.

$$\Theta(\theta) = 0$$

The trivial solution is obtained, so there are no negative eigenvalues. According to the principle of superposition, the general solution to the PDE is a linear combination of the eigenfunctions over all the eigenvalues.

$$\boxed{u(r,\theta) = \sum_{n=1}^{\infty} B_n \left(\frac{r^{2n}}{a^{2n}} - \frac{a^{2n}}{r^{2n}}\right) \sin 2n\theta}$$

Use the boundary condition at $r = b$ to determine the coefficients B_n.

$$u(b,\theta) = \sum_{n=1}^{\infty} B_n \left(\frac{b^{2n}}{a^{2n}} - \frac{a^{2n}}{b^{2n}}\right) \sin 2n\theta = f(\theta)$$

Multiply both sides by $\sin 2p\theta$, where p is an integer.

$$\sum_{n=1}^{\infty} B_n \left(\frac{b^{2n}}{a^{2n}} - \frac{a^{2n}}{b^{2n}}\right) \sin 2n\theta \sin 2p\theta = f(\theta) \sin 2p\theta$$

Integrate both sides with respect to θ from 0 to $\pi/2$.

$$\int_0^{\pi/2} \left[\sum_{n=1}^{\infty} B_n \left(\frac{b^{2n}}{a^{2n}} - \frac{a^{2n}}{b^{2n}}\right) \sin 2n\theta \sin 2p\theta\right] d\theta = \int_0^{\pi/2} f(\theta) \sin 2p\theta \, d\theta$$

Split up the integral on the left and bring the constants in front.

$$\sum_{n=1}^{\infty} B_n \left(\frac{b^{2n}}{a^{2n}} - \frac{a^{2n}}{b^{2n}}\right) \int_0^{\pi/2} \sin 2n\theta \sin 2p\theta \, d\theta = \int_0^{\pi/2} f(\theta) \sin 2p\theta \, d\theta$$

Because the sine functions are orthogonal, this integral on the left is zero if $n \neq p$. Only if $n = p$ does the integral yield a nonzero result.

$$B_n \left(\frac{b^{2n}}{a^{2n}} - \frac{a^{2n}}{b^{2n}}\right) \int_0^{\pi/2} \sin^2 2n\theta \, d\theta = \int_0^{\pi/2} f(\theta) \sin 2n\theta \, d\theta$$

Evaluate the integral.

$$B_n \left(\frac{b^{4n} - a^{4n}}{a^{2n}b^{2n}}\right) \left(\frac{\pi}{4}\right) = \int_0^{\pi/2} f(\theta) \sin 2n\theta \, d\theta$$

Therefore,

$$\boxed{B_n = \frac{4a^{2n}b^{2n}}{\pi(b^{4n} - a^{4n})} \int_0^{\pi/2} f(\theta) \sin 2n\theta \, d\theta.}$$

Part (b)

Substitute the product solution into the homogeneous boundary conditions.

$$\begin{aligned} u(r,0)=0 &\quad\to\quad R(r)\Theta(0)=0 &\quad\to\quad \Theta(0)=0 \\ u(a,\theta)=0 &\quad\to\quad R(a)\Theta(\theta)=0 &\quad\to\quad R(a)=0 \\ u(b,\theta)=0 &\quad\to\quad R(b)\Theta(\theta)=0 &\quad\to\quad R(b)=0 \end{aligned}$$

Solve the ODE for R.

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right)=\lambda$$

Check to see if zero is an eigenvalue: $\lambda = 0$.

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right)=0$$

Multiply both sides by R/r.

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right)=0$$

Integrate both sides with respect to r.

$$r\frac{dR}{dr}=C_1$$

Divide both sides by r.

$$\frac{dR}{dr}=\frac{C_1}{r}$$

Integrate both sides with respect to r once more.

$$R(r)=C_1\ln r+C_2$$

Apply the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} R(a)&=C_1\ln a+C_2=0 \\ R(b)&=C_1\ln b+C_2=0 \end{aligned}$$

Solving this system of equations yields $C_1 = 0$ and $C_2 = 0$.

$$R(r)=0$$

The trivial solution is obtained, so zero is not an eigenvalue. Suppose now that $\lambda \neq 0$.

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right)=\lambda$$

Expand the left side.

$$\frac{r}{R}(R'+rR'')=\lambda$$

Multiply both sides by R and bring all terms to the left side.

$$r^2R''+rR'-\lambda R=0$$

This is an equidimensional ODE, so its solutions are of the form $R(r) = r^m$.

$$R = r^m \quad \rightarrow \quad R' = mr^{m-1} \quad \rightarrow \quad R'' = m(m-1)r^{m-2}$$

Substitute these formulas into the ODE and solve the resulting equation for m.

$$r^2 m(m-1)r^{m-2} + rmr^{m-1} - \lambda r^m = 0$$

$$m(m-1)r^m + mr^m - \lambda r^m = 0$$

$$m(m-1) + m - \lambda = 0$$

$$m^2 - \lambda = 0$$

$$m = \{\pm\sqrt{\lambda}\}$$

Check for positive eigenvalues: $\lambda = \mu^2$.

$$m = \{\pm\mu\}$$

Then two solutions to the ODE for R are $R = r^{-\mu}$ and $R = r^{\mu}$. By the principle of superposition, the general solution is a linear combination of these two.

$$R(r) = C_3 r^{-\mu} + C_4 r^{\mu}$$

Apply the boundary conditions to determine C_3 and C_4.

$$R(a) = C_3 a^{-\mu} + C_4 a^{\mu} = 0$$
$$R(b) = C_3 b^{-\mu} + C_4 b^{\mu} = 0$$

No value of μ can satisfy these equations, so $C_3 = 0$ and $C_4 = 0$.

$$R(r) = 0$$

The trivial solution is obtained, so there are no positive eigenvalues. Check for negative eigenvalues: $\lambda = -\gamma^2$.

$$m = \{\pm i\gamma\}$$

Then two solutions to the ODE for R are $R = r^{-i\gamma}$ and $R = r^{i\gamma}$. By the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned}
R(r) &= C_5 r^{-i\gamma} + C_6 r^{i\gamma} \\
&= C_5 e^{\ln r^{-i\gamma}} + C_6 e^{\ln r^{i\gamma}} \\
&= C_5 e^{-i\gamma \ln r} + C_6 e^{i\gamma \ln r} \\
&= C_5[\cos(\gamma \ln r) - i\sin(\gamma \ln r)] + C_6[\cos(\gamma \ln r) + i\sin(\gamma \ln r)] \\
&= (C_5 + C_6)\cos(\gamma \ln r) + (-iC_5 + iC_6)\sin(\gamma \ln r) \\
&= C_7 \cos(\gamma \ln r) + C_8 \sin(\gamma \ln r)
\end{aligned}$$

Apply the boundary conditions to determine C_7 and C_8.

$$R(a) = C_7 \cos(\gamma \ln a) + C_8 \sin(\gamma \ln a) = 0$$
$$R(b) = C_7 \cos(\gamma \ln b) + C_8 \sin(\gamma \ln b) = 0$$

Solve this first equation for C_7

$$C_7 = -C_8 \frac{\sin(\gamma \ln a)}{\cos(\gamma \ln a)}$$

and substitute it into the second equation.

$$\left[-C_8 \frac{\sin(\gamma \ln a)}{\cos(\gamma \ln a)}\right] \cos(\gamma \ln b) + C_8 \sin(\gamma \ln b) = 0$$

Assume that $C_8 \neq 0$ and divide both sides by C_8.

$$-\frac{\sin(\gamma \ln a)}{\cos(\gamma \ln a)} \cos(\gamma \ln b) + \sin(\gamma \ln b) = 0$$

Multiply both sides by $\cos(\gamma \ln a)$.

$$\sin(\gamma \ln b)\cos(\gamma \ln a) - \sin(\gamma \ln a)\cos(\gamma \ln b) = 0$$

$$\sin(\gamma \ln b - \gamma \ln a) = 0$$

$$\sin\left(\gamma \ln \frac{b}{a}\right) = 0$$

$$\gamma \ln \frac{b}{a} = n\pi, \quad n = 1, 2, \ldots$$

$$\gamma = \frac{n\pi}{\ln \frac{b}{a}}$$

There are negative eigenvalues $\lambda = -\left(\frac{n\pi}{\ln \frac{b}{a}}\right)^2$, and the eigenfunctions associated with them are

$$\begin{aligned}
R(r) &= C_7 \cos(\gamma \ln r) + C_8 \sin(\gamma \ln r) \\
&= \left[-C_8 \frac{\sin(\gamma \ln a)}{\cos(\gamma \ln a)}\right] \cos(\gamma \ln r) + C_8 \sin(\gamma \ln r) \\
&= \frac{C_8}{\cos(\gamma \ln a)} [\sin(\gamma \ln r)\cos(\gamma \ln a) - \sin(\gamma \ln a)\cos(\gamma \ln r)] \\
&= \frac{C_8}{\cos(\gamma \ln a)} \sin(\gamma \ln r - \gamma \ln a) \\
&= \frac{C_8}{\cos(\gamma \ln a)} \sin\left(\gamma \ln \frac{r}{a}\right) \quad \rightarrow \quad R_n(r) = \sin\left(\frac{n\pi}{\ln \frac{b}{a}} \ln \frac{r}{a}\right).
\end{aligned}$$

With $\lambda = -\left(\frac{n\pi}{\ln \frac{b}{a}}\right)^2$, solve the ODE for Θ.

$$\Theta'' = \left(\frac{n\pi}{\ln \frac{b}{a}}\right)^2 \Theta$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$\Theta(\theta) = C_9 \cosh\left(\frac{n\pi}{\ln \frac{b}{a}}\theta\right) + C_{10} \sinh\left(\frac{n\pi}{\ln \frac{b}{a}}\theta\right)$$

Use the boundary condition $\Theta(0) = 0$ to determine one of the constants.

$$\Theta(0) = C_9 = 0$$

The Θ-eigenfunction is then

$$\Theta(\theta) = C_{10} \sinh\left(\frac{n\pi}{\ln\frac{b}{a}}\theta\right).$$

According to the principle of superposition, the general solution to the PDE is a linear combination of the eigenfunctions $R_n(r)\Theta_n(\theta)$ over all the eigenvalues.

$$\boxed{u(r,\theta) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{\ln\frac{b}{a}}\theta\right) \sin\left(\frac{n\pi}{\ln\frac{b}{a}}\ln\frac{r}{a}\right)}$$

Apply the final boundary condition to determine the coefficients A_n.

$$u\left(r, \frac{\pi}{2}\right) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{\ln\frac{b}{a}}\frac{\pi}{2}\right) \sin\left(\frac{n\pi}{\ln\frac{b}{a}}\ln\frac{r}{a}\right) = f(r)$$

Multiply both sides by $\frac{1}{r}\sin\left(\frac{p\pi}{\ln\frac{b}{a}}\ln\frac{r}{a}\right)$. This $1/r$ factor is included to account for the logarithm in the sine's argument.

$$\sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi^2}{2\ln\frac{b}{a}}\right) \frac{1}{r}\sin\left(\frac{n\pi}{\ln\frac{b}{a}}\ln\frac{r}{a}\right) \sin\left(\frac{p\pi}{\ln\frac{b}{a}}\ln\frac{r}{a}\right) = \frac{f(r)}{r}\sin\left(\frac{p\pi}{\ln\frac{b}{a}}\ln\frac{r}{a}\right)$$

Integrate both sides with respect to r from a to b.

$$\int_a^b \left[\sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi^2}{2\ln\frac{b}{a}}\right) \frac{1}{r}\sin\left(\frac{n\pi}{\ln\frac{b}{a}}\ln\frac{r}{a}\right) \sin\left(\frac{p\pi}{\ln\frac{b}{a}}\ln\frac{r}{a}\right)\right] dr = \int_a^b \frac{f(r)}{r}\sin\left(\frac{p\pi}{\ln\frac{b}{a}}\ln\frac{r}{a}\right) dr$$

Split up the integrals and bring the constants in front.

$$\sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi^2}{2\ln\frac{b}{a}}\right) \int_a^b \frac{1}{r}\sin\left(\frac{n\pi}{\ln\frac{b}{a}}\ln\frac{r}{a}\right) \sin\left(\frac{p\pi}{\ln\frac{b}{a}}\ln\frac{r}{a}\right) dr = \int_a^b \frac{f(r)}{r}\sin\left(\frac{p\pi}{\ln\frac{b}{a}}\ln\frac{r}{a}\right) dr$$

Because the sine functions are orthogonal, this integral on the left is zero if $n \neq p$. Only if $n = p$ does the integral yield a nonzero result.

$$A_n \sinh\left(\frac{n\pi^2}{2\ln\frac{b}{a}}\right) \int_a^b \frac{1}{r}\sin^2\left(\frac{n\pi}{\ln\frac{b}{a}}\ln\frac{r}{a}\right) dr = \int_a^b \frac{f(r)}{r}\sin\left(\frac{n\pi}{\ln\frac{b}{a}}\ln\frac{r}{a}\right) dr$$

Evaluate the integral.

$$A_n \sinh\left(\frac{n\pi^2}{2\ln\frac{b}{a}}\right)\left(\frac{1}{2}\ln\frac{b}{a}\right) = \int_a^b \frac{f(r)}{r}\sin\left(\frac{n\pi}{\ln\frac{b}{a}}\ln\frac{r}{a}\right) dr$$

Therefore,

$$\boxed{A_n = \frac{2}{\sinh\left(\frac{n\pi^2}{2\ln\frac{b}{a}}\right)\ln\frac{b}{a}} \int_a^b \frac{f(r)}{r}\sin\left(\frac{n\pi}{\ln\frac{b}{a}}\ln\frac{r}{a}\right) dr.}$$